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Virasoro Characters from Bethe Equations for the Critical Ferromagnetic Three-State Potts Model

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We obtain new fermionic sum representations for the Virasoro characters of the conformal field theory describing the ferromagnetic three-state Potts spin chain. These arise from the fermionic quasiparticle excitations derived from the Bethe equations for the eigenvalues of the Hamiltonian. In the conformal scaling limit, the Bethe equations provide a description of the spectrum in terms of one genuine quasiparticle and two "ghost" excitations with a limited microscopic momentum range. This description is reflected in the structure of the character formulas, and suggests a connection with the integrable perturbation of dimensions $(2/3, 2/3)^+$ which breaks the S_3 symmetry of the conformal field theory down to $\mathbf{Z}_2$.

KEY WORDS: Three-state Potts; Bethe equations; conformal field theory; quasiparticles; affine Lie algebras; Virasoro characters.

1. INTRODUCTION

The critical three-state Potts model was found to be integrable over 20 years ago[1,2] and since these initial investigations it has been the subject of many studies.[3] Recently[4] it was shown that the order-one excitations of the antiferromagnetic three-state Potts spin chain,[5] computed from the formalism of functional and Bethe equations,[6-12] can be used to construct expressions for the characters of the conformal field theory of $\mathbf{Z}_4$-parafermions. Since these equations yield excitations which obey a fermionic exclusion rule, we call these fermionic sum representations. These character formulas were previously obtained by Lepowsky and Primc[13] from considerations of the representation theory of the affine Lie algebra

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$A_1^{(1)}$. The characters, which in this case are branching functions of $(A_1^{(1)})_4/U(1)$, are the building blocks of the modular invariant partition function of the conformal field theory.

Here we provide a parallel discussion for the ferromagnetic chain, leading to fermionic sum representations for the Virasoro characters[14] of the $\mathbf{Z}_3$-parafermionic conformal field theory which is associated with this model.[15–17] These representations, which we will now summarize, are quite different from the ones of ref. 13.

The normalized Virasoro characters $\hat{\chi}_\Delta \equiv q^{1/30-\Delta}\chi_\Delta$ of the $\mathbf{Z}_3$-parafermionic conformal field theory, with central charge $c = 4/5$ and conformal dimensions $\Delta = \Delta_{r,s} = [(6r-5s)^2 - 1]/120$ $(r = 1, 2, 3, 4,\ s = 1, 3, 5)$, are given by[14]

$$\hat{\chi}_{\Delta_{r,s}}(q) = \hat{\chi}_{\Delta_{5-r,6-s}}(q) = \frac{1}{(q)_\infty} \sum_{k=-\infty}^{\infty} [q^{k(30k+6r-5s)} - q^{(5k+r)(6k+s)}] \tag{1.1}$$

Our result here is that these characters can be written in the form

$$\hat{\chi}_\Delta(q) = \sum_{\substack{m_1, m_2, m_3 = 0 \\ \text{restrictions}}}^{\infty} q^{\frac{1}{4}\mathbf{m} C_{A_3} \mathbf{m}^t - \frac{1}{2} L(\mathbf{m})} \frac{1}{(q)_{m_1}} \times \begin{bmatrix} \frac{1}{2}(m_1 + m_3 + u_2) \\ m_2 \end{bmatrix}_q \begin{bmatrix} \frac{1}{2}(m_2 + u_3) \\ m_3 \end{bmatrix}_q \tag{1.2}$$

where $(q)_0 = 1$, $(q)_m = \prod_{a=1}^{m} (1 - q^a)$, the q-binomial coefficient is defined for integer m, n as

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{cases} (q)_n/(q)_{n-m}\,(q)_m & \text{if } n \geqslant m \geqslant 0 \\ 0 & \text{otherwise} \end{cases} \tag{1.3}$$

$\mathbf{m} = (m_1, m_2, m_3)$, and C_{A_3} is the Cartan matrix of the Lie algebra A_3:

$$C_{A_3} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \tag{1.4}$$

The restrictions on the integers m_a in Eq. (1.2) depend on the character in question, and are such that m_a are either even (e) or odd (o). These restrictions are listed, together with the u_a and the linear translation terms $L(\mathbf{m})$, in Table I. We note that for characters other than $\hat{\chi}_0$ and $\hat{\chi}_3$ there is more than one representation of the form (1.2), and that the formulas corresponding to lines (1)–(7), (9), and (12)–(13) in the table are special cases of

Table I. Restrictions and Linear Translation Terms for the Characters $\hat{\chi}_\Delta$ in Eq. (2)[a]

	Δ	m_1	m_2	m_3	u_2	u_3	$L(\mathbf{m})$
(1)	0	e	e	e	0	0	0
(2)	2/5	o	e	e	1	0	1
(3)		o	o	o	0	1	1
(4)	7/5	e	e	o	1	0	3
(5)		e	o	e	0	1	3
(6)	3	o	e	o	0	0	6
(7)	1/15	o	e	o	2	0	m_2
(8)		e	e	e	2	0	m_2
(9)		e	o	o	1	1	m_2
(10)		o	o	e	1	1	m_2
(11)		{e	o	o	1	-1	m_1-m_3
		+o	o	e}			
(12)	2/3	e	e	o	1	0	m_2+1
(13)		o	e	e	1	0	m_2+1
(14)		{e	o	e	0	-1	m_1-m_3+1
		+o	o	o}			

[a] Here e ≡ even and o ≡ odd. Note that the characters $\hat{\chi}_{1/15}$ and $\hat{\chi}_{2/3}$ have a two-term expression as well as one-term expressions.

the fermionic sum representations for Virasoro characters presented already in ref. 18.

The modular invariant partition function of the conformal field theory associated with the three-state Potts model is written in a factorized form in terms of these characters[17]:

$$(q\bar{q})^{-1/30}\hat{Z} = [\chi_0(q)+\chi_3(q)][\chi_0(\bar{q})+\chi_3(\bar{q})] + [\chi_{2/5}(q)+\chi_{7/5}(q)][\chi_{2/5}(\bar{q})+\chi_{7/5}(\bar{q})] + 2\chi_{1/15}(q)\,\chi_{1/15}(\bar{q})+2\chi_{2/3}(q)\,\chi_{2/3}(\bar{q}) \qquad (1.5)$$

Here the variable q $(=\bar{q})$ is associated with contributions coming from right- (left-) moving excitations, as discussed in Section 2.

In this paper we construct a direct connection between the low-lying spectrum of the spin chain Hamiltonian and the conformal field theory. We do so by computing the partition function of the spin chain in an appropriate scaling limit [see (1.12) below], obtaining expressions of the form (1.2) for the Virasoro characters. Our starting point is the quasiparticle nature of the spectrum.

A many-body system is said to have a quasiparticle spectrum if in the infinite-size limit the energy E and momentum P of the low-lying excitations above the ground state are of the form

$$E - E_{GS} = \sum_{\alpha,\,\text{rules}} \sum_{j=1}^{m_\alpha} e_\alpha(P_j^\alpha), \quad P - P_{GS} \equiv \sum_{\alpha,\,\text{rules}} \sum_{j=1}^{m_\alpha} P_j^\alpha (\text{mod } 2\pi) \tag{1.6}$$

where m_α is the number of excitations of type α in a given state. The rules of composition in (1.6) depend on the model in question, and commonly include a fermionic exclusion rule

$$P_j^\alpha \neq P_k^\alpha \qquad \text{if} \quad j \neq k \tag{1.7}$$

in which case the spectrum is said to be fermionic.

There are many cases where the excitation spectrum is gapless, i.e., one or more of the $e_\alpha(P^\alpha)$ vanish at some value of the momentum, say at $P^\alpha = 0$, and

$$e_\alpha(P^\alpha) \sim v_\alpha |P^\alpha| \qquad \text{for} \quad P^\alpha \sim 0 \tag{1.8}$$

where $v_\alpha > 0$ is the Fermi velocity of the excitation of type α.

The partition function of the quantum spin chain at temperature T is the sum over all states,

$$Z = \sum_{\{\text{states}\}} e^{-E/k_B T} = e^{-E_{GS}/k_B T} \sum_{\{\text{states}\}} e^{-(E - E_{GS})/k_B T} \tag{1.9}$$

and the specific heat in the thermodynamic limit is defined by

$$C(T) = -T \frac{\partial^2 f}{\partial T^2}, \qquad \text{where} \quad f = -k_B T \lim_{M \to \infty} \frac{1}{M} \ln Z \tag{1.10}$$

Here M is the size of the system and the temperature T has some fixed positive value. When the spectrum is of the form (1.8), at low temperature the specific heat is dominated by quasiparticle states (1.6) with vanishing single-particle energies and exhibits a linear T behavior. Therefore, in order to extract this behavior it is necessary to consider only excitations of this type in the sum over states (1.9). We refer to the resulting partition function, in the limit $M \to \infty$ and with the ground-state energy factored out, as the conformal partition function. More explicitly, the conformal field theory partition function (1.5) is obtained from

$$\hat{Z} = \lim e^{E_{GS}/k_B T} Z \tag{1.11}$$

in the limit

$$T \to 0 \quad \text{and} \quad M \to \infty, \quad \text{with } MT \text{ fixed} \tag{1.12}$$

Using (1.6) and (1.8), we see that $\hat{Z}$ is a function of the variable

$$q \equiv \exp\left(-\frac{2\pi v}{Mk_{\mathrm{B}}T}\right) \tag{1.13}$$

If there are no additional length scales in the problem, the $q \to 1$ behavior of $\hat{Z}$ and the $T \to 0$ limit of the partition function in the thermodynamic limit (1.10) should match. Indeed, the leading $q \to 1$ behavior of $\hat{Z}$ was computed in ref. 18 from the expression for the characters (1.2), where it was shown that the linear behavior of the specific heat obtained in this way is the same as that obtained in the thermodynamic limit at low temperature.[7,19]

We also remark that the connection between these two different computations goes beyond giving just the same final result for the value of the specific heat coefficient. In the analysis of the $q \to 1$ behavior of sums generalizing (1.2) for characters of a large class of conformal field theories, one encounters[18] the same equations (involving dilogarithms) which appear in thermodynamic Bethe Ansatz analyses of the corresponding spin chains, as well as of factorizable scattering theories that are associated with certain integrable perturbations of the conformal field theory in question. We will say more about the relation between fermionic character sums and integrable perturbations in Section 5.

In ref. 4 it was shown for the antiferromagnetic three-state Potts chain that the sum over low-lying excitations with a massless dispersion relation (1.8) gives rise to the D_4 (ref. 20) modular-invariant partition functions of the $\mathbf{Z}_4$-parafermionic conformal field theory. In that model there are three different excitations, all having the same linear dispersion relation. In contrast, the spectrum of the ferromagnetic three-state Potts chain has a different structure. While there is only one type of quasiparticle excitation of the kind found for the antiferromagnetic case,[5] there are two more excitations, which do not contribute to the energies at order one ($=M^0$), but rather determine the degeneracy of states of the order-one excitation spectrum,[21] thus affecting the thermodynamics through entropy considerations. In the calculation of the partition function, where we take the energy of all excitations to have a linear dispersion relation, this can be viewed as a statement that the momentum range of these latter two excitations is microscopic (of order M^{-1}), instead of being macroscopic (order M^0) as it is for the quasiparticle excitation.

In Section 2 we define the model and introduce the relevant Bethe equations, as well as the order-one spectrum. In Section 3 we use the finite-size studies of refs. 21 and 22 to extend the order-one analysis of the spectrum[5] to order $1/M$, and study the sectors of the partition function which give rise to the representations (1), (2), (4), and (6) in Table I for the characters $\hat{\chi}_0$, $\hat{\chi}_3$, $\hat{\chi}_{2/5}$, and $\hat{\chi}_{7/5}$. The sector of the partition function which corresponds to the character $\hat{\chi}_{1/15}$ is analyzed in Section 4. This gives a representation for $\hat{\chi}_{1/15}$ in terms of five sums of the form (1.2). In Section 5 we contrast the form (1.2) with the result of ref. 13 and discuss the relation of these different fermionic representations for the conformal field theory characters to certain integrable massive extensions.

2. THE GAPLESS THREE-STATE POTTS CHAIN

The gapless three-state Potts quantum spin chain of M sites with periodic boundary conditions is defined by the Hamiltonian

$$H = \pm \frac{2}{\sqrt{3}} \sum_{j=1}^{M} \{X_j + X_j^\dagger + Z_j Z_{j+1}^\dagger + Z_j^\dagger Z_{j+1}\} \tag{2.1}$$

where $Z_{M+1} = Z_1$ and for $j = 1,..., M$ the matrices X_j and Z_j are written as a direct product of M 3×3 matrices:

$$X_j = I \otimes I \otimes \cdots \otimes \underbrace{X}_{j\text{th}} \otimes \cdots \otimes I, \qquad Z_j = I \otimes I \otimes \cdots \otimes \underbrace{Z}_{j\text{th}} \otimes \cdots \otimes I \tag{2.2}$$

Here I is the identity matrix and

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \qquad \omega = e^{2\pi i/3} \tag{2.3}$$

The Hamiltonian with the $(+)-$ sign is referred to as the (anti-) ferromagnetic spin chain. The Hamiltonian has a $\mathbf{Z}_3$ spin-rotation invariance and thus the $\mathbf{Z}_3$ charges $Q = 0, \pm 1$ are good quantum numbers. In addition, (2.1) is invariant under complex conjugation and hence the sectors $Q = \pm 1$ have equal eigenvalues and in $Q = 0$ the eigenvalue $C = \pm 1$ of the charge conjugation operator is a good quantum number.

The Hamiltonian (2.1) is derived from the two-dimensional critical three-state Potts model of classical statistical mechanics. The eigenvalues of the transfer matrix of the latter model satisfy functional equations[11] which, when specialized to the Hamiltonian point,[21] yield equations for the eigenvalues of the Hamiltonian. These eigenvalues are given by

$$E = \sum_{j=1}^{L} \cot\left(i\lambda_j + \frac{\pi}{12}\right) - \frac{2M}{\sqrt{3}}, \qquad L = 2(M - |Q|), \quad Q = 0, \pm 1 \tag{2.4}$$

where the rapidities λ_j satisfy a set of equations of the form of Bethe equations:

$$\left[\frac{\sinh(i\pi/12 - \lambda_j)}{\sinh(i\pi/12 + \lambda_j)}\right]^{2M} = (-1)^{M+1} \prod_{k=1}^{L} \frac{\sinh[i\pi/3 - (\lambda_j - \lambda_k)]}{\sinh[i\pi/3 + (\lambda_j - \lambda_k)]}, \qquad j = 1,..., L \tag{2.5}$$

The corresponding momentum, which is defined as the eigenvalue of the translation operator, is given by

$$e^{iP} = \prod_{k=1}^{L} \frac{\sinh(\lambda_k - i\pi/12)}{\sinh(\lambda_k + i\pi/12)} \tag{2.6}$$

The solutions of the Bethe equations are sets of (possibly complex) roots $\{\lambda_j\}$, and in the large-lattice limit each root belongs to one of five different classes,[21] the roots in each class having a fixed value of the imaginary part of λ_j,[3]

$$\lambda_j \quad \text{is called} \quad \left\{\begin{matrix} \lambda_j^+ \\ \lambda_j^- \\ \lambda_j^{2s} \\ \lambda_j^{-2s} \\ \lambda_j^{ns} \end{matrix}\right\} \quad \text{if} \quad \Im\mathrm{m}(\lambda_j) = \left\{\begin{matrix} 0 \\ \pi/2 \\ \pm\pi/6 \\ \pm\pi/3 \\ \pm\pi/4 \end{matrix}\right\} \tag{2.7}$$

The last three classes of roots occur in complex conjugate pairs, and are referred to as complex pairs. We define m_α (where $\alpha = +, -, 2s, -2s, ns$) to be the number of roots in each class, complex pairs being counted once. A detailed analysis of Eqs. (2.5) was performed in ref. 21. We summarize those results of that paper which we will use here in the Appendix.

The order-one excitation spectrum obtained from (2.4)–(2.5) in the limit $M \to \infty$ was found in ref. 5. It was shown there that for the ferro-

[3] Note that the definition of λ here has a factor of $-1/2$ relative to the definition in ref. 21.

magnetic case, the order-one energy gaps can all be written in the quasi-particle form

$$E - E_{\mathrm{GS}} = \sum_{j=1}^{m_+} e_+(P_j^+) \tag{2.8}$$

where $m_+ = 2m_{ns} + 3m_- + 4m_{-2s}$ and the single-particle energy is

$$e_+(P_j^+) = 6\sin(|P_j^+|/2), \qquad 0 \leqslant P_j^+ \leqslant 2\pi \tag{2.9}$$

so that the Fermi velocity is $v = 3$. The momentum of a single-particle state is expressed in terms of its rapidity λ^+ as

$$P(\lambda^+) \equiv \pi + 4\tan^{-1}(\tanh 3\lambda^+) \quad (\mathrm{mod}\ 2\pi) \tag{2.10}$$

The number of states characterized by the same set $\{P_j^+\}$ [and thus by the corresponding set of single-particle energies $\{e_+(P_j^+)\}$] is, in the sector $Q = 0$,[21]

$$\binom{m_- + m_{-2s}}{m_{-2s}}\binom{2m_- + 2m_{-2s} + m_{ns}}{m_{ns}} \tag{2.11}$$

where $\binom{a}{b}$ is the binomial coefficient. This stems from the fact that the other excitations (ns, $-2s$) carry no energy, yet states differing only in their content of $\{\lambda_j^\alpha\}_{\alpha = ns, -2s}$ have to be counted individually.

In order to construct the scaled partition function (1.11) of the model, we extend the order-one spectrum to momenta near zero. At such momenta, the energy is

$$e_+(P^+) = \begin{cases} vP^+ & \text{for} \quad P^+ \sim 0 \\ v(2\pi - P^+) & \text{for} \quad P^+ \sim 2\pi \end{cases} \tag{2.12}$$

Note that there are no absolute value signs, and the momentum is no longer defined mod 2π. This amounts to extending the order-one result (2.9) to order $1/M$; however, at this order we must consider two additional contributions to the energy:

1. To order one, the excitations ns and $-2s$ contribute zero energy. However, to order $1/M$ they may carry energy, and indeed we find that $e_\alpha(P^\alpha) = vP^\alpha$ for $\alpha = ns, -2s$, but with P^α restricted to only a microscopic range, of order $1/M$. Here v is the same as in Eq. (2.12).

2. Constant (independent of momentum or the number of excitations) contributions of order $1/M$ to the energy must be accounted for. These contributions, which give the conformal dimensions $\Delta_{r,s}$, have been computed from functional equations for the transfer matrix by Klümper and Pearce.[22]

From Eq. (2.6) and Eqs. (A.18), (A.23), and (A.26) of the Appendix, we see that the total momentum of any state can be written as

$$P=\frac{2\pi}{M}\left\{\sum_{j=1}^{m_+}\bar{I}_j^+ + \sum_{j=1}^{m_{-2s}} I_j^{-2s} + \sum_{j=1}^{m_{ns}} I_j^{ns} + L(m_\alpha)\right\} \tag{2.13}$$

where $L(m_\alpha)$ is some linear shift which depends on the sector under consideration. In Eq. (2.12) the energy depends linearly on the P_j^α, which are quantized in units of $2\pi/M$ and are directly related to the (half-) integers of the logarithmic Bethe equations (A.1) as

$$P_j^+=\frac{2\pi}{M}I_j^+ + \pi \equiv \frac{2\pi}{M}\bar{I}_j^+, \qquad P_j^{-2s}=\frac{2\pi}{M}I_j^{-2s}, \qquad P_j^{ns}=\frac{2\pi}{M}I_j^{ns} \tag{2.14}$$

The energy can thus be expressed in terms of the I_j^α.

The spectrum (and so the partition function) splits into different sectors of definite $\mathbf{Z}_3$ charge $Q=0$, ± 1, and furthermore the sector $Q=0$ splits into subsectors of parity number $C=\pm 1$, corresponding to m_- being even ($C=1$) or odd ($C=-1$). Hence we can discuss separately each sector, which give rise to different characters, as in the antiferromagnetic case.[4]

3. THE CHARACTERS IN THE SECTOR $Q=0$

The (half-) integers in this sector are chosen from the ranges (A.8)–(A.9), and hence we see from (2.14) that the P_j^α are chosen from the ranges of spacing $2\pi/M$ with the following limits:

$$-\frac{2\pi}{M}\left[\frac{1}{2}(m_- + m_{-2s} - 1)\right] \leqslant P_j^+ \leqslant 2\pi + \frac{2\pi}{M}\left[\frac{1}{2}(m_- + m_{-2s} - 1)\right] \tag{3.1}$$

$$-\frac{2\pi}{M}\left[\frac{1}{2}(m_- + m_{-2s} - 1)\right] \leqslant P_j^{-2s} \leqslant \frac{2\pi}{M}\left[\frac{1}{2}(m_- + m_{-2s} - 1)\right] \tag{3.2}$$

$$-\frac{2\pi}{M}\left[\frac{1}{2}(2m_- + 2m_{-2s} + m_{ns} - 1)\right]$$

$$\leqslant P_j^{ns} \leqslant \frac{2\pi}{M}\left[\frac{1}{2}(2m_- + 2m_{-2s} + m_{ns} - 1)\right] \tag{3.3}$$

As is the case for the excitations of the antiferromagnetic chain,[4] the range of single-particle momenta for the "+"excitations is macroscopic: it is of order 2π for any finite m_α in the limit $M \to \infty$. In contrast, the ranges for P_j^{ns} and P_j^{-2s} are of order $1/M$ and allow only a finite number of states in the limit $M \to \infty$, for given m_α. We refer to excitations with such microscopic momentum ranges as "ghost" excitations.

One expects that the partition function factorizes into right- and left-moving contributions, as in the antiferromagnetic case, so that the characters of the model are obtained by considering these contributions separately. However, in the ferromagnetic case only the P_j^+ can be considered to be right- or left-moving, where right- (left-) movers indicates $P_j^+ \sim 0$ ($P_j^+ \sim 2\pi$).

When taking the limit $M \to \infty$, right- (left-) movers can be considered to lie on a semi-infinite range, since the range for P^+ is macroscopic, allowing for an infinite number of momentum states. Therefore, we rewrite the momentum range for right-movers in this limit as

$$-\frac{2\pi}{M}\left[\frac{1}{2}(m_- + m_{-2s} - 1)\right] \leqslant P_j^+ < \infty \qquad \text{for right-movers} \tag{3.4}$$

replacing Eq. (3.1). For the left-movers it is convenient to replace P^+ by $P^+ - 2\pi$, so that the momentum range in the $M \to \infty$ limit is

$$-\infty < P_j^+ \leqslant \frac{2\pi}{M}\left[\frac{1}{2}(m_- + m_{-2s} - 1)\right] \qquad \text{for left-movers} \tag{3.5}$$

and the dispersion relation (2.12) now reads

$$e_+(P^+) = \begin{cases} vP^+ & \text{for right-movers} \\ -vP^+ & \text{for left-movers} \end{cases} \tag{3.6}$$

There are four characters corresponding to the $Q = 0$ sector (since there is a symmetry between right- and left-movers, below we restrict our attention to the right-movers):

1. The vacuum character $\hat{\chi}_0$, which corresponds to the sector of the partition function with only right-movers and positive parity, $C = +1$.
2. The character $\hat{\chi}_3$, which corresponds to the sector with only right-movers and negative parity, $C = -1$.

3. The character $\hat{\chi}_{2/5}$, which has one left-mover and the rest right-movers, with $C = +1$.
4. $\hat{\chi}_{7/5}$, which has only one left-mover and $C = -1$.

We will discuss in detail the construction of the partition function in the sector corresponding to $\hat{\chi}_0$, item 1 above, and then outline the computation of the other characters.

3.1. Construction of the Character $\hat{\chi}_0$

The sector of the partition function (1.9) which has only right-movers and $Q = 0$, $C = +1$ is computed as follows. The excitation energy is simply the sum over the individual excitations near $P^\alpha \sim 0$,

$$E - E_{\mathrm{GS}} = v \left\{ \sum_{j=1}^{m_+} P_j^+ + \sum_{j=1}^{m_{-2s}} P_j^{-2s} + \sum_{j=1}^{m_{ns}} P_j^{ns} \right\} \tag{3.7}$$

The partition function is the sum over all right-moving excitations with momentum ranges (3.2)–(3.4), subject to the fermionic exclusion rule (1.7), and the restriction that m_- be even. In Table II we show the lowest energy states in this sector. The general expression for the partition function in this sector is

$$\hat{Z}_0 = \sum_{\{\text{states}\}} e^{-vP/k_B T} = \sum_{\{I_j^\alpha\}} q^{(\sum_j I_j^+ + \sum_j I_j^{-2s} + \sum_j I_j^{ns})} \tag{3.8}$$

with q defined as in Eq. (1.13). Here the I_j^α are restricted as in Eq. (A.8) in the Appendix with m_- even. As in (3.4), in the limit $M \to \infty$ we have for right-moving "+"-excitations

$$-\tfrac{1}{2}(m_- + m_{-2s} - 1) \leqslant \bar{I}_j^+ < \infty \tag{3.9}$$

The restrictions on the integers are implemented by using two integer partitions, $Q_m(N; n)$ and $Q_m(N) \equiv Q_m(N; \infty)$, where $Q_m(N; n)$ is the number of partitions of $N \geqslant 0$ into m distinct nonnegative integers each less than or equal to n. The partition function (3.8) subject to the restrictions (3.4), (3.2), (3.3) then becomes

$$\begin{aligned}\hat{Z}_0 = &\sum_{\substack{m_-, m_{-2s}, m_{ns} = 0 \\ m_- \text{ even} \\ m_+ = 2m_{ns} + 3m_- + 4m_{-2s}}}^{\infty} \sum_{N_+, N_{-2s}, N_{ns} = 0}^{\infty} Q_{m_+}(N_+)\, q^{N_+ - \frac{1}{2}m_+(m_{-2s} + m_- - 1)} \\ &\times Q_{m_{-2s}}(N_{-2s}; m_{-2s} + m_- - 1)\, q^{N_{-2s} - \frac{1}{2}m_{-2s}(m_{-2s} + m_- - 1)} \\ &\times Q_{m_{ns}}(N_{ns}; m_{ns} + 2m_{-2s} + 2m_- - 1)\, q^{N_{ns} - \frac{1}{2}m_{ns}(m_{ns} + 2m_{-2s} + 2m_- - 1)}\end{aligned} \tag{3.10}$$

Table II. The First Terms of the Partition Function in the Sector $Q=0$ and $C=1$, Where m_- Is Even, and $m_+ = 2m_{ns} + 3m_- + 4m_{-2s}$[a]

Order	m_+	m_{-2s}	m_-	m_{ns}	$P^{+,-2s}_{min}$	P^{ns}_{min}	$[P^{ns}; P^{-2s}; P^+]$ (units of π/M)	N	Tot
q^0	0	0	0	0	—	—	[—; —; —]	1	1
q^2	2	0	0	1	π/M	0	[0; —; 1, 3]	1	1
q^3	2	0	0	1	π/M	0	[0; —; 1, 5]	1	1
q^4	2	0	0	1	π/M	0	[0; —; 1, 7], [0; —; 3, 5]	2	2
q^5	2	0	0	1	π/M	0	[0; —; 1, 9], [0; —; 3, 7]	2	2
q^6	2	0	0	1	π/M	0	[0; —; 1, 11], [0; —; 3, 9], [0; —; 5, 7]	3	
	4	1	0	0	0	—	[—; 0; 0, 2, 4, 6]	1	4
q^7	2	0	0	1	π/M	0	[0; —; 1, 13], [0; —; 3, 11], [0; —; 5, 9]	3	
	4	1	0	0	0	—	[—; 0; 0, 2, 4, 8]	1	4
q^8	2	0	0	1	π/M	0	[0; —; 1, 15], [0; —; 3, 13], [0; —; 5, 11], [0; —; 7, 9]	4	
	4	1	0	0	0	—	[—; 0; 0, 2, 4, 10], [—; 0; 0, 2, 6, 8]	2	
	4	0	0	2	π/M	$-\pi/M$	[−1, 1; —; 1, 3, 5, 7]	1	7
q^9	2	0	0	1	π/M	0	[0; —; 1, 17], [0; —; 3, 15], [0; —; 5, 13], [0; —; 7, 11]	4	
	4	1	0	0	0	—	[—; 0; 0, 2, 4, 12], [—; 0; 0, 2, 6, 10]		
							[—; 0; 0, 4, , 8]	3	
	4	0	0	2	π/M	$-\pi/M$	[−1; 1; —; 1, 3, 5, 9]	1	8
q^{10}	2	0	0	1	π/M	0	[0; —; 1, 19], [0; —; 3, 17],		
							[0; —; 5, 15], [0; —; 7, 13], [0; —; 9, 11]	5	
	4	1	0	0	0	—	[—; 0; 0, 2, 4, 14], [—; 0; 0, 2, 6, 12]		
							[—; 0; 0, 4, 6, 10], [—; 0; 0, 2, 8, 10]		
							[—; 0; 2, 4, 6, 8]	5	
	4	0	0	2	π/M	$-\pi/M$	[−1; 1; —; 1, 3, 5, 11], [−1, 1; —; 1, 3, 7, 9]	2	12

q^{11}	2	0	0	1	π/M	0	[0; —; 1, 21], [0; —; 3, 19], [0; —; 5, 17], [0; —; 7, 15], [0; —; 9, 13]	5	
	4	1	0	0	0	—	[—; 0; 0, 2, 4, 16], [—; 0; 0, 2, 6, 14], [—; 0; 0, 2, 8, 12], [—; 0; 0, 4, 6, 12], [—; 0; 0, 4, 8, 10], [—; 0; 2, 4, 6, 10]	6	
	4	0	0	2	π/M	$-\pi/M$	[−1; 1; —; 1, 3, 5, 13], [−1, 1; —; 1, 3, 7, 11], [−1, 1; —; 1, 5, 7, 9]	3	14
q^{12}	2	0	0	1	π/M	0	[0; —; 1, 23], [0; —; 3, 21], [0; —; 5, 19], [0; —; 7, 17], [0; —; 9, 15], [0; —; 11, 13]	6	
	4	1	0	0	0	—	[—; 0; 0, 2, 4, 18], [—; 0; 0, 2, 6, 16], [—; 0; 0, 2, 8, 14], [—; 0; 0, 2, 10, 12], [—; 0; 0, 4, 6, 14], [—; 0; 2, 4, 6, 12], [—; 0; 0, 4, 8, 12], [—; 0; 2, 4, 8, 10], [—; 0; 0, 6, 8, 10]	9	
	4	0	0	2	π/M	$-\pi/M$	[−1, 1; —; 1, 3, 5, 15], [−1, 1; —; 1, 3, 7, 13], [−1, 1; —; 1, 5, 7, 11], [−1, 1; —; 1, 3, 9, 11], [−1, 1; —; 3, 5, 7, 9]	5	
	6	0	2	0	$-\pi/M$	—	[—; —; −1, 1, 3, 5, 7, 9]	1	21

[a] The momentum ranges are given in Eqs. (3.2)–(3.4). The sum of the momenta in the square brackets gives the total momentum, and thus the power of q, listed on the left. N is the number of states with given m_α and fixed total momentum, whose overall number is listed on the right. These are the coefficients of q^n in the power expansion of $\hat{\chi}_0$.

Table III. The First Terms in the Partition Function in the Sector $Q=0$ and $C=-1$, Corresponding to m_- Odd, and $m_+ = 2m_{ns} + 3m_- + 4m_{-2s}$[a]

Order	m_+	m_{-2s}	m_-	m_{ns}	$P^{+,-2s}_{min}$	P^{ns}_{min}	$[P^{ns}; P^{-2s}; P^+]$ (units of π/M)	N	Tot
q^3	3	0	1	0	0	—	[—; —; 0, 2, 4]	1	1
q^4	3	0	1	0	0	—	[—; —; 0, 2, 6]	1	1
q^5	3	0	1	0	0	—	[—; —; 0, 2, 8], [—; —; 0, 4, 6]	2	2
q^6	3	0	1	0	0	—	[—; —; 0, 2, 10], [—; —; 0, 4, 8], [—; —; 2, 4, 6]	3	3
q^7	3	0	1	0	0	—	[—; —; 0, 2, 12], [—; —; 0, 4, 10], [—; —; 0, 6, 8], [—; —; 2, 4, 8]	4	4
q^8	3	0	1	0	0	—	[—; —; 0, 2, 14], [—; —; 0, 4, 12], [—; —; 0, 6, 10], [—; —; 2, 4, 10], [—; —; 2, 6, 8]	5	5
q^9	3	0	1	0	0	—	[—; —; 0, 2, 16], [—; —; 0, 4, 14], [—; —; 2, 4, 12], [—; —; 0, 6, 12], [—; —; 2, 6, 10], [—; —; 4, 6, 8], [—; —; 0, 8, 10]	7	
	5	0	1	1	0	$-2\pi/M$	[−2; —; 0, 2, 4, 6, 8]	1	8
q^{10}	3	0	1	0	0	—	[—; —; 0, 2, 18], [—; —; 0, 4, 16], [—; —; 2, 4, 14], [—; —; 0, 6, 14], [—; —; 2, 6, 12], [—; —; 4, 6, 10], [—; —; 0, 8, 12], [—; —; 2, 8, 10]	8	
	5	0	1	1	0	$-2\pi/M$	[−2; —; 0, 2, 4, 6, 10] [0; —; 0, 2, 4, 6, 8]	2	10

q^{11}	3	0	1	0	0	—	[—; —; 0, 2, 20], [—; —; 0, 4, 18], [—; —; 2, 4, 16], [—; —; 0, 6, 16], [—; —; 2, 6, 14], [—; —; 4, 6, 12], [—; —; 0, 8, 14], [—; —; 2, 8, 12], [—; —; 4, 8, 10], [—; —; 0, 10, 12]	10	
	5	0	1	1	0	$-2\pi/M$	[−2; —; 0, 2, 4, 6, 12], [−2; —; 0, 2, 4, 8, 10], [0; —; 0, 2, 4, 6, 10], [2; —; 0, 2, 4, 6, 8]	4	14
q^{12}	3	0	1	0	0	—	[—; —; 0, 2, 22], [—; —; 0, 4, 20], [—; —; 2, 4, 18], [—; —; 0, 6, 18], [—; —; 2, 6, 16], [—; —; 4, 6, 14], [—; —; 0, 8, 16], [—; —; 2, 8, 14], [—; —; 4, 8, 12], [—; —; 0, 10, 14], [—; —; 2, 10, 12], [—; —; 6, 8, 10]	12	
	5	0	1	1	0	$-2\pi/M$	[−2; —; 0, 2, 4, 6, 14], [−2; —; 0, 2, 4, 8, 12], [−2; —; 0, 2, 6, 8, 10], [0; —; 0, 2, 4, 6, 12], [0; —; 0, 2, 4, 8, 10], [2; —; 0, 2, 4, 6, 10]	6	18

[a] The momentum ranges are the same as in Table II. The total number of states on the right corresponds to the first few terms in the expansion of $q^3\tilde{\chi}_3$.

The exponents of q above are essentially the total momenta of each type of excitation, i.e., the sums over the integers $N_\alpha = \sum_j I_j^\alpha$. The partitions count the number of times $q^{\sum_\alpha N_\alpha}$ occurs in the partition function, which is the number of ways N_α can be divided between m_α fermionic excitations.

The sum (3.10) can be reexpressed using the identity[23,24]

$$\sum_{N=0}^{\infty} Q_m(N; n)\, q^N = q^{m(m-1)/2} \begin{bmatrix} n+1 \\ m \end{bmatrix}_q \tag{3.11}$$

which, when $n \to \infty$, reduces to

$$\sum_{N=0}^{\infty} Q_m(N)\, q^N = \frac{q^{m(m-1)/2}}{(q)_m} \tag{3.12}$$

Using these identities, (3.10) becomes

$$\begin{aligned} \hat{Z}_0 = & \sum_{\substack{m_-, m_{-2s}, m_{ns} = 0 \\ m_+ = 2m_{ns} + 3m_- + 4m_{-2s} \\ m_- \text{ even}}}^{\infty} q^{-\frac{1}{2}(m_+ + m_{-2s})(m_{-2s} + m_- - 1) - \frac{1}{2}m_{ns}(m_{ns} + 2m_{-2s} + 2m_- - 1)} \\ & \times q^{\frac{1}{2}m_+(m_+ - 1) + \frac{1}{2}m_{-2s}(m_{-2s} - 1) + \frac{1}{2}m_{ns}(m_{ns} - 1)} \\ & \times \frac{1}{(q)_{m_+}} \begin{bmatrix} m_{-2s} + m_- \\ m_{-2s} \end{bmatrix}_q \begin{bmatrix} m_{ns} + 2m_{-2s} + 2m_- \\ m_{ns} \end{bmatrix}_q \end{aligned} \tag{3.13}$$

The form of the sum may be further simplified by changing variables to

$$m_1 = m_+, \qquad m_2 = 2m_- + 2m_{-2s}, \qquad m_3 = m_- \tag{3.14}$$

which results in the expression (1.2) with the restriction that all m_a are even and $L(\mathbf{m}) = 0$, $u_a = 0$. This is the expression listed on line (1) of Table I.

This expression is quite different in form from the one given in (1.1). Nevertheless we find that

$$\hat{Z}_0 = \hat{\chi}_0 \tag{3.15}$$

This has been verified as an equality between the series expansions of the two expressions to order q^{200}, using Mathematica.

3.2. Construction of $\hat{\chi}_3$

In Table III we present the lowest energy states of the sector $C = -1$, where all m_+ are right-movers. The calculation of the partition function $\hat{Z}_3$ is identical to that of the last section, except that now m_- is odd in Eqs.

Table IV. The First Terms for the Sector of the Partition Function Corresponding to $Q=0$ and $C=1$, Where One of the "+"-Excitations Is a Left-Mover, and the Rest Are Right-Movers[a]

Order	m_+	m_{-2s}	m_-	m_{ns}	$P_{\min}^{+,-2s}$	$P_{\min}^{ns}$	$[P^{ns}; P^{-2s}; P^+]$ (units of π/M)	Shift	N	Tot
q^0	1	0	0	1	π/M	0	[0; —; 1]	$-\pi/M$	1	1
q^1	1	0	0	1	π/M	0	[0; —; 3]	$-\pi/M$	1	1
q^2	1	0	0	1	π/M	0	[0; —; 5]	$-\pi/M$	1	1
q^3	1	0	0	1	π/M	0	[0; —; 7]	$-\pi/M$	1	
	3	1	0	0	0	—	[—; 0; 0, 2, 4]	0	1	2
q^4	1	0	0	1	π/M	0	[0; —; 9]	$-\pi/M$	1	
	3	1	0	0	0	—	[—; 0; 0, 2, 6]	0	1	
	3	0	0	2	π/M	$-\pi/M$	[−1, 1; —; 1, 3, 5]	$-\pi/M$	1	3
q^5	1	0	0	1	π/M	0	[0; —; 11]	$-\pi/M$	1	
	3	1	0	0	0	—	[—; 0; 0, 2, 8], [—; 0; 0, 4, 6]	0	2	
	3	0	0	2	π/M	$-\pi/M$	[−1, 1; —; 1, 3, 7]	$-\pi/M$	1	4
q^6	1	0	0	1	π/M	0	[0; —; 13]	$-\pi/M$	1	
	3	1	0	0	0	—	[—; 0; 0, 2, 10], [—; 0; 0, 4, 8], [—; 0; 2, 4, 6]	0	3	
	3	0	0	2	π/M	$-\pi/M$	[−1, 1; —; 1, 3, 9], [−1, 1; —; 1, 5, 7]	$-\pi/M$	2	6
q^7	1	0	0	1	π/M	0	[0; —; 15]	$-\pi/M$	1	
	3	1	0	0	0	—	[—; 0; 0, 2, 12], [—; 0; 0, 4, 10], [—; 0; 2, 4, 8], [—; 0; 0, 6, 8]	0	4	
	3	0	0	2	π/M	$-\pi/M$	[−1, 1; —; 1, 3, 11], [−1, 1; —; 1, 5, 9] [−1, 1; —; 3, 5, 7]	$-\pi/M$	3	8
q^8	1	0	0	1	π/M	0	[0; —; 17]	$-\pi/M$	1	
	3	1	0	0	0	—	[—; 0; 0, 2, 14], [—; 0; 0, 4, 12], [—; 0; 2, 4, 10], [—; 0; 0, 6, 10], [—; 0; 2, 6, 8]	0	5	
	3	0	0	2	π/M	$-\pi/M$	[−1, 1; —; 1, 3, 13], [−1, 1; —; 1, 5, 11], [−1, 1; —; 3, 5, 9], [−1, 1; —; 1, 7, 9]	$-\pi/M$	4	
	5	0	2	0	$-\pi/M$	—	[—; —; −1, 1, 3, 5, 7]	π/M	1	11
q^9	1	0	0	1	π/M	0	[0; —; 19]	$-\pi/M$	1	
	3	1	0	0	0	—	[—; 0; 0, 2, 16], [—; 0; 0, 4, 14], [—; 0; 2, 4, 12], [—; 0; 0, 6, 12], [—; 0; 2, 6, 10], [—; 0; 4, 6, 8], [—; 0; 0, 8, 10],	0	7	
	3	0	0	2	π/M	$-\pi/M$	[−1, 1; —; 1, 3, 15], [−1, 1; —; 1, 5, 13], [−1, 1; —; 3, 5, 11], [−1, 1; —; 1, 7, 11], [−1, 1; —; 3, 7, 9]	$-\pi/M$	5	
	5	0	2	0	$-\pi/M$	—	[—; —; −1, 1, 3, 5, 9]	π/M	1	
	5	1	0	1	0	$-2\pi/M$	[−2; 0; 0, 2, 4, 6, 8]	0	1	15

[a] This corresponds to m_- even and $m_+ = 2m_{ns} + 3m_- + 4m_{-2s} - 1$. The momentum ranges are the same as in Table II, and there is an additional term ("shift") in the momentum of $(\pi/M)(m_- + m_{-2s} - 1)$ which is the momentum of the left-mover. The coefficients on the right correspond to the expansion of $\hat{\chi}_{2/5}$.

Table V. The First Terms in the Partition Function in the Sector $Q = 0$, $C = -1$, Where One "+"-Excitation Is Left-Moving and All the Rest are Right-Movers[a]

Order	m_+	m_{-2s}	m_-	m_{ns}	$P_{\min}^{+,-2s}$	$P_{\min}^{ns}$	$[P^{ns}; P^{-2s}; P^+]$ (units of π/M)	Shift	N	Tot
q^1	2	0	1	0	0	—	[—; —; 0, 2]	0	1	1
q^2	2	0	1	0	0	—	[—; —; 0, 4]	0	1	1
q^3	2	0	1	0	0	—	[—; —; 0, 6], [—; —; 2, 4]	0	2	2
q^4	2	0	1	0	0	—	[—; —; 0, 8], [—; —; 2, 6]	0	2	2
q^5	2	0	1	0	0	—	[—; —; 0, 10], [—; —; 2, 8], [—; —; 4, 6]	0	3	
	4	0	1	1	0	$-2\pi/M$	[−2; —; 0, 2, 4, 6]	0	1	4
q^6	2	0	1	0	0	—	[—; —; 0, 12], [—; —; 2, 10], [—; —; 4, 8]	0	3	
	4	0	1	1	0	$-2\pi/M$	[−2; —; 0, 2, 4, 8], [0; —; 0, 2, 4, 6]	0	2	5
q^7	2	0	1	0	0	—	[—; —; 0, 14], [—; —; 2, 12], [—; —; 4, 10], [—; —; 6, 8]	0	4	
	4	0	1	1	0	$-2\pi/M$	[−2; —; 0, 2, 4, 10], [−2, —; 0, 2, 6, 8], [0; —; 0, 2, 4, 8], [2; —; 0, 2, 4, 6]	0	4	8
q^8	2	0	1	0	0	—	[—; —; 0, 16], [—; —; 2, 14], [—; —; 4, 12], [—; —; 6, 10]	0	4	
	4	0	1	1	0	$-2\pi/M$	[−2; —; 0, 2, 4, 12], [−2; —; 0, 2, 6, 10], [−2; —; 0, 4, 6, 8], [0; —; 0, 2, 4, 10], [0; —; 0, 2, 6, 8], [2; —; 0, 2, 4, 8]	0	6	10
q^9	2	0	1	0	0	—	[—; —; 0, 18], [—; —; 2, 16], [—; —; 4, 14], [—; —; 6, 12], [—; —; 8, 10]	0	5	
	4	0	1	1	0	$-2\pi/M$	[−2; —; 0, 2, 4, 14], [−2; —; 0, 2, 6, 12], [−2; —; 0, 4, 6, 10], [−2; —; 2, 4, 6, 8], [−2; —; 0, 2, 8, 10], [0; —; 0, 2, 4, 12], [0; —; 0, 2, 6, 10], [0; —; 0, 4, 6, 8], [2; —; 0, 2, 4, 10], [2; —; 0, 2, 6, 8]	0	10	15
q^{10}	2	0	1	0	0	—	[—; —; 0, 20], [—; —; 2, 18], [—; —; 4, 16], [—; —; 6, 14], [—; —; 8, 12]	0	5	
	4	0	1	1	0	$-2\pi/M$	[−2; —; 0, 2, 4, 16], [−2; —; 0, 2, 6, 14], [−2; —; 0, 4, 6, 12], [−2; —; 2, 4, 6, 10], [−2; —; 0, 2, 8, 12], [−2; —; 0, 4, 8, 10], [0; —; 0, 2, 4, 14], [0; —; 0, 2, 6, 12], [0; —; 0, 4, 6, 10], [0; —; 2, 4, 6, 8], [0; —; 0, 2, 8, 10], [2; —; 0, 2, 4; 12], [2; —; 0, 2, 6, 10], [2; —; 0, 4, 6, 8]	0	14	19

[a] The shift and m_+ are as in Table IV, and the momentum ranges are as in Table II. The coefficients on the right correspond to the expansion of

(3.10) and (3.13). Using series expansions, we verify that the resulting expression $\hat{Z}_3$ is equal to $q^3\hat{\chi}_3$ to order q^{200}. With the change of variables (3.14), this results in the expression on line (6) of Table I.

3.3. Construction of $\hat{\chi}_{2/5}$ and $\hat{\chi}_{7/5}$

The characters $\hat{\chi}_{2/5}$ and $\hat{\chi}_{7/5}$ occur when one of the m_+ is a left-mover, and all the rest are right-movers. This amounts to setting $m_+ = 2m_{ns} + 3m_- + 4m_{-2s} - 1$ in the partition sum (3.10) and (3.13). Also, there is an additive term to the momentum of the form $(\pi v/M)(m_- + m_{ns} - 1)$, which is the lowest energy state of the single left-moving "+"-excitation allowed by Eq. (3.5). The character $\hat{\chi}_{2/5}$ occurs for $C = +1$, i.e., m_- even, and the character $\hat{\chi}_{7/5}$ occurs for $C = -1$, m_- odd. We tabulate the lowest energy states for these two sectors in Tables IV and V. The expression for the resulting partition functions is

$$\begin{aligned}\hat{Z}_{2/5(7/5)} = &\sum_{\substack{m_+, m_{-2s}, m_{ns} = 0 \\ m_+ = 2m_{ns} + 3m_- + 4m_{-2s} - 1 \\ m_- \text{ even (odd)}}}^{\infty} \sum_{N_x = 0}^{\infty} q^{\frac{1}{2}(m_- + m_{-2s} - 1)} \\ &\times Q_{m_+}(N_+)\, q^{N_+ - \frac{1}{2}m_+(m_{-2s} + m_- - 1)} \\ &\times Q_{m_{-2s}}(N_{-2s}; m_{-2s} + m_- - 1)\, q^{N_{-2s} - \frac{1}{2}m_{-2s}(m_{-2s} + m_- - 1)} \\ &\times Q_{m_{ns}}(N_{ns}; m_{ns} + 2m_{-2s} + 2m_- - 1)\, q^{N_{ns} - \frac{1}{2}m_{ns}(m_{ns} + 2m_{-2s} + 2m_- - 1)}\end{aligned} \tag{3.16}$$

Using the identities (3.11)–(3.12) and the change of variables (3.14), we bring $\hat{Z}_{2/5}$ and $q^{-1}\hat{Z}_{7/5}$ to the form (1.2) with the restrictions listed on lines (2) and (4) of Table I, i.e., we find that

$$\hat{Z}_{2/5} = \hat{\chi}_{2/5}, \qquad \hat{Z}_{7/5} = q\hat{\chi}_{7/5} \tag{3.17}$$

The other expressions for the two characters $\hat{\chi}_{2/5}$ and $\hat{\chi}_{7/5}$, corresponding to lines (3) and (5) of that table, are conjectured forms. Using a power series expansion, we showed that all these forms are equal to the corresponding expressions (1.1) for $\hat{\chi}_{2/5}$ and $\hat{\chi}_{7/5}$, to order q^{200}.

4. THE SECTOR $Q = 1$

The analysis of this sector is more involved than for the $Q = 0$ sector, since (see Appendix) there are five different subsectors to be considered, where the integers range over different intervals. Each of these subsectors

Table VI. The First Terms in the Partition Function for the Sector $Q=1$ and $m_- - m_{++} = 1$[a]

Order	m_+	m_{-2s}	m_-	m_{ns}	$P_{\min}^{+;-2s}$	$P_{\min}^{ns}$	$[P^{ns}; P^{-2s}; P^+]$ (units of π/M)	N	Tot
q^1	1	0	1	0	$2\pi/M$	—	[—; —; 2]	1	1
q^2	1	0	1	0	$2\pi/M$	—	[—; —; 4]	1	1
q^3	1	0	1	0	$2\pi/M$	—	[—; —; 6]	1	1
q^4	1	0	1	0	$2\pi/M$	—	[—; —; 8]	1	1
q^5	1	0	1	0	$2\pi/M$	—	[—; —; 10]	1	1
q^6	1	0	1	0	$2\pi/M$	—	[—; —; 12]	1	
	3	0	1	1	$2\pi/M$	0	[0; —; 2, 4, 6]	1	2
q^7	1	0	1	0	$2\pi/M$	—	[—; —; 14]	1	
	3	0	1	1	$2\pi/M$	0	[0; —; 2, 4, 8]	1	2
q^8	1	0	1	0	$2\pi/M$	—	[—; —; 16]	1	
	3	0	1	1	$2\pi/M$	0	[0; —; 2, 4, 10], [0; —; 2, 6, 8]	2	
	4	0	2	0	π/M	—	[—; —; 1, 3, 5, 7]	1	4
q^9	1	0	1	0	$2\pi/M$	—	[—; —; 18]	1	
	3	0	1	1	$2\pi/M$	0	[0; —; 2, 4, 12], [0; —; 2, 6, 10], [0; —; 4, 6, 8]	3	
	4	0	2	0	π/M	—	[—; —; 1, 3, 5, 9]	1	5
q^{10}	1	0	1	0	$2\pi/M$	—	[—; —; 20]	1	
	3	0	1	1	$2\pi/M$	0	[0; —; 2, 4, 14], [0; —; 2, 6, 12], [0; —; 4, 6, 10], [0; —; 2, 8, 10]	4	
	4	0	2	0	π/M	—	[—; —; 1, 3, 5, 11], [—; —; 1, 3, 7, 9]	2	7
q^{11}	1	0	1	0	$2\pi/M$	—	[—; —; 22]	1	
	3	0	1	1	$2\pi/M$	0	[0; —; 2, 4, 16], [0; —; 2, 6, 14], [0; —; 4, 6, 12], [0; —; 2, 8, 12], [0; —; 4, 8, 10]	5	
	4	0	2	0	π/M	—	[—; —; 1, 3, 5, 13], [—; —; 1, 3, 7, 11] [—; —; 1, 5, 7, 9]	3	9
q^{12}	1	0	1	0	$2\pi/M$	—	[—; —; 24]	1	
	3	0	1	1	$2\pi/M$	0	[0; —; 2, 4, 18], [0; —; 2, 6, 16], [0; —; 4, 6, 14], [0; —; 2, 8, 14], [0; —; 4, 8, 12], [0; —; 2, 10, 12], [0; —; 6, 8, 10]	7	
	4	0	2	0	π/M	—	[—; —; 1, 3, 5, 15], [—; —; 1, 3, 7, 13], [—; —; 1, 5, 7, 11], [—; —; 1, 3, 9, 11], [—; —; 3, 5, 7, 9]	5	13

[a] Here $m_+ = 2m_{ns} + 3m_- + 4m_{-2s} - 2$, and the momentum ranges are: $-(\pi/M)(m_{-2s}+m_- - 3) \leqslant P_j^+ < \infty$, $-(\pi/M)(m_{-2s}+m_- - 3) \leqslant P_j^{-2s} \leqslant (\pi/M)(m_{-2s}+m_- - 3)$, $-(\pi/M)(m_{ns}+2m_{-2s}+2m_- - 3) \leqslant P_j^{ns} \leqslant (\pi/M)(m_{ns}+2m_{-2s}+2m_- - 3)$.

Table VII. The First Terms in the Partition Function for the Sector $Q=1$ and $m_- - m_{++} = -1$[a]

Order	m_+	m_{-2s}	m_-	m_{ns}	$P_{\min}^{+,-2s}$	$P_{\min}^{ns}$	$[P^{ns}; P^{-2s}; P^+]$ (units of π/M)	N	Tot
q^2	2	0	0	0	π/M	—	[—; —; 1, 3]	1	1
q^3	2	0	0	0	π/M	—	[—; —; 1, 5]	1	1
q^4	2	0	0	0	π/M	—	[—; —; 1, 7], [—; —; 3, 5]	2	2
q^5	2	0	0	0	π/M	—	[—; —; 1, 9], [—; —; 3, 7]	2	2
q^6	2	0	0	0	π/M	—	[—; —; 1, 11], [—; —; 3, 9], [—; —; 5, 7]	3	3
q^7	2	0	0	0	π/M	—	[—; —; 1, 13], [—; —; 3, 11], [—; —; 5, 9]	3	
	4	0	0	1	π/M	$-2\pi/M$	[−2; —; 1, 3, 5, 7]	1	4
q^8	2	0	0	0	π/M	—	[—; —; 1, 15], [—; —; 3, 13], [—; —; 5, 11], [—; —; 7, 9]	4	
	4	0	0	1	π/M	$-2\pi/M$	[−2; —; 1, 3, 5, 9], [0, —, 1, 3, 5, 7]	2	6
q^9	2	0	0	0	π/M	—	[—; —; 1, 17], [—; —; 3, 15], [—; —; 5, 13], [—; —; 7, 11]	4	
	4	0	0	1	π/M	$-2\pi/M$	[−2; —; 1, 3, 5, 11], [−2, —; 1, 3, 7, 9], [0; —; 1, 3, 5, 9], [2; —; 1, 3, 5, 7]	4	8
q^{10}	2	0	0	0	π/M	—	[—; —; 1, 19], [—; —; 3, 17], [—; —; 5, 15], [—; —; 7, 13], [—; —; 9, 11]	5	
	4	0	0	1	π/M	$-2\pi/M$	[−2; —; 1, 3, 5, 13], [−2; —; 1, 3, 7, 11], [−2; —; 1, 3, 7, 9], [0; —; 1, 3, 5, 9], [0; —; 1, 3, 7, 9], [2; —; 1, 3, 5, 9]	6	
	5	0	1	0	0	—	[—; —; 0, 2, 4, 6, 8]	1	12
q^{11}	2	0	0	0	π/M	—	[—; —; 1, 21], [—; —; 3, 19], [—; —; 5, 17], [—; —; 7, 15], [—; —; 9, 13]	5	
	4	0	0	1	π/M	$-2\pi/M$	[−2; —; 1, 3, 5, 15], [−2; —; 1, 3, 7, 13], [−2; —; 1, 5, 7, 11], [−2; —; 3, 5, 7, 9], [−2; —; 1, 3, 9, 11], [0; —; 1, 3, 5, 13], [0; —; 1, 3, 7, 11], [0; —; 1, 5, 7, 9], [2; —; 1, 3, 5, 11], [2; —; 1, 3, 7, 9]	10	
	5	0	1	0	0	—	[—; —; 0, 2, 4, 6, 10]	1	16
q^{12}	2	0	0	0	π/M	—	[—; —; 1, 23], [—; —; 3, 21], [—; —; 5, 19], [—; —; 7, 17], [—; —; 9, 15], [—; —; 11, 13]	6	
	4	0	0	1	π/M	$-2\pi/M$	[−2; —; 1, 3, 5, 17], [−2; —; 1, 3, 7, 15], [−2; —; 1, 5, 7, 13], [−2; —; 3, 5, 7, 11], [−2; —; 1, 3, 9, 13], [−2; —; 1, 5, 9, 11], [0; —; 1, 3, 5, 15], [0; —; 1, 3, 7, 13], [0; —; 1, 5, 7, 11], [0; —; 3, 5, 7, 9], [0; —; 1, 3, 9, 11], [2; —; 1, 3, 5, 13], [2; —; 1, 3, 7, 11], [2; —; 1, 5, 7, 9]	14	
	5	0	1	0	0	—	[—; —; 0, 2, 4, 6, 12], [—; —; 0, 2, 4, 8, 10]	2	22

[a] Here $m_+ = 2m_{ns} + 3m_- + 4m_{-2s} + 2$, and the momentum ranges are: $-(\pi/M)(m_{-2s} + m_- - 1) \leqslant P_j^+ < \infty$, $-(\pi/M)(m_{-2s} + m_- - 1) \leqslant P_j^{-2s} \leqslant (\pi/M)(m_{-2s} + m_- - 1)$, $-(\pi/M)(m_{ns} + 2m_{-2s} + 2m_- - 1) \leqslant P_j^{ns} \leqslant (\pi/M)(m_{ns} + 2m_{-2s} + 2m_- - 1)$.

gives rise to a separate sum in the sector of the partition function corresponding to right-moving excitations. The resulting five sums add together to form the Virasoro character $\hat{\chi}_{1/15}$, as we now describe in more detail.

The momentum ranges are given by the integer ranges (A.8) and (A.10)–(A.14) in the Appendix, and the relations (A.18), (A.23), and (A.26) of the total momentum to the integers. We will present the computation for each subsector separately, where for each subsector all "+"-excitations are right-movers.

1. $m_- - m_{++} = +1$: Here we see from Eq. (A.6) that $m_+ = 2m_{ns} + 3m_- + 4m_{-2s} - 2$, and the lowest energy states are shown in Table VI, where the integer range is that of Eq. (A.10). The partition sum starts with $m_- = 1$, since $m_- > m_{++} \geqslant 0$:

$$\hat{Z}^{(1)}_{1/15} = \sum_{m_-=1}^{\infty} \sum_{\substack{m_{ns}, m_{-2s}=0 \\ m_+ = 2m_{ns} + 3m_- + 4m_{-2s} - 2}}^{\infty} q^{\frac{1}{2}m_+(m_+ - 1) + \frac{1}{2}m_+(3 - m_- - m_{-2s})} \frac{1}{(q)_{m_+}}$$
$$\times q^{\frac{1}{2}m_{-2s}(m_{-2s} - 1) + \frac{1}{2}m_{-2s}(3 - m_- - m_{-2s})} \begin{bmatrix} m_{-2s} + m_- - 2 \\ m_{-2s} \end{bmatrix}_q$$
$$\times q^{\frac{1}{2}m_{ns}(m_{ns} - 1) + \frac{1}{2}m_{ns}(3 - 2m_- - 2m_{-2s} - m_{ns})} \begin{bmatrix} m_{ns} + 2m_{-2s} + 2m_- - 2 \\ m_{ns} \end{bmatrix}_q \tag{4.1}$$

2. $m_- - m_{++} = -1$: From the sum rule (A.6) we see that $m_+ = 2m_{ns} + 3m_- + 4m_{-2s} + 2$, and the integer range is given by (A.11). The lowest energy states are shown in Table VII, and the general expression for the partition function is

$$\hat{Z}^{(2)}_{1/15} = \sum_{\substack{m_-, m_{ns}, m_{-2s}=0 \\ m_+ = 2m_{ns} + 3m_- + 4m_{-2s} + 2}}^{\infty} q^{\frac{1}{2}m_+(m_+ - 1) + \frac{1}{2}m_+(1 - m_- - m_{-2s})} \frac{1}{(q)_{m_+}}$$
$$\times q^{\frac{1}{2}m_{-2s}(m_{-2s} - 1) + \frac{1}{2}m_{-2s}(1 - m_- - m_{-2s})} \begin{bmatrix} m_{-2s} + m_- \\ m_{-2s} \end{bmatrix}_q$$
$$\times q^{\frac{1}{2}m_{ns}(m_{ns} - 1) + \frac{1}{2}m_{ns}(-1 - 2m_- - 2m_{-2s} - m_{ns})}$$
$$\times \begin{bmatrix} m_{ns} + 2m_{-2s} + 2m_- + 2 \\ m_{ns} \end{bmatrix}_q \tag{4.2}$$

3. $m_- = m_{++} = 0$: Since $m_- = 0$, there are no $-2s$ excitations. The relevant momentum range is obtained from the integer range (A.12), and

the lowest energy states are listed in Table VIII. The general expression for the partition function is a simple sum over *ns* excitations, with $m_+ = 2m_{ns}$:

$$\hat{Z}^{(3)}_{1/15} = \sum_{\substack{m_{ns}=0 \\ m_+ = 2m_{ns}}}^{\infty} \frac{q^{m_+(m_+ + 1)/2}}{(q)_{m_+}} = \sum_{m_{ns}=0}^{\infty} \frac{q^{m_{ns}(2m_{ns}+1)}}{(q)_{2m_{ns}}} \tag{4.3}$$

4. $m_- = m_{++} \neq 0$: There are two subsectors with this characteristic, corresponding to the integer ranges (A.13) and (A.14). These integer ranges are asymmetric, so there is a shift term in the total momentum, as shown in (A.23) and (A.26), of the form $\mp(2\pi v/M)(\frac{1}{2}m_{ns} + m_- + m_{-2s})$. For both of these sectors $m_+ = 2m_{ns} + 3m_- + 4m_{-2s}$. The lowest energy states are listed in Tables IX and X. The sums take the forms, for the integer range (A.13),

$$\begin{aligned}\hat{Z}^{(4)}_{1/15} = \sum_{m_-=1}^{\infty} \sum_{\substack{m_{ns}, m_{-2s}=0 \\ m_+ = 2m_{ns}+3m_- + 4m_{-2s}}}^{\infty} & q^{-(m_- + m_{-2s} + \frac{1}{2}m_{ns})} \\ & \times q^{\frac{1}{2}m_+(m_+ - 1) + \frac{1}{2}m_+(3 - m_- - m_{-2s})} \frac{1}{(q)_{m_+}} \\ & \times q^{\frac{1}{2}m_{-2s}(m_{-2s} - 1) + \frac{1}{2}m_{-2s}(1 - m_- - m_{-2s})} \begin{bmatrix} m_{-2s} + m_- \\ m_{-2s} \end{bmatrix}_q \\ & \times q^{\frac{1}{2}m_{ns}(m_{ns} - 1) + \frac{1}{2}m_{ns}(2m_- - 2m_{-2s} - m_{ns})} \begin{bmatrix} m_{ns} + 2m_{-2s} + 2m_- \\ m_{ns} \end{bmatrix}_q \end{aligned} \tag{4.4}$$

and for the integer range (A.14)

$$\begin{aligned}\hat{Z}^{(5)}_{1/15} = \sum_{m_-=1}^{\infty} \sum_{\substack{m_{ns}, m_{-2s}=0 \\ m_+ = 2m_{ns}+3m_- + 4m_{-2s}}}^{\infty} & q^{(m_- + m_{-2s} + \frac{1}{2}m_{ns})} \\ & \times q^{\frac{1}{2}m_+(m_+ - 1) + \frac{1}{2}m_+(1 - m_- - m_{-2s})} \frac{1}{(q)_{m_+}} \\ & \times q^{\frac{1}{2}m_{-2s}(m_{-2s} - 1) + \frac{1}{2}m_{-2s}(3 - m_- - m_{-2s})} \begin{bmatrix} m_{-2s} + m_- \\ m_{-2s} \end{bmatrix}_q \\ & \times q^{\frac{1}{2}m_{ns}(m_{ns} - 1) + \frac{1}{2}m_{ns}(2 - 2m_- - 2m_{-2s} - m_{ns})} \begin{bmatrix} m_{ns} + 2m_{-2s} + 2m_- \\ m_{ns} \end{bmatrix}_q \end{aligned} \tag{4.5}$$

Table VIII. The First Terms in the Sector of the Partition Function for the Sector $Q=1$ and $m_- = m_{++} = 0$[a]

Order	m_+	m_{-2s}	m_-	m_{ns}	$P_{min}^{+,-2s}$	P_{min}^{ns}	$[P^{ns}; P^{-2s}; P^+]$ (units of π/M)	N	Tot
q^0	0	0	0	0	—	—	[—; —; —]	1	1
q^3	2	0	0	1	$2\pi/M$	0	[0; —; 2, 4]	1	1
q^4	2	0	0	1	$2\pi/M$	0	[0; —; 2, 6]	1	1
q^5	2	0	0	1	$2\pi/M$	0	[0; —; 2, 8], [0; —; 4, 6]	2	2
q^6	2	0	0	1	$2\pi/M$	0	[0; —; 2, 10], [0; —; 4, 8]	2	2
q^7	2	0	0	1	$2\pi/M$	0	[0; —; 2, 12], [0; —; 4, 10], [0; —; 6, 8]	3	3
q^8	2	0	0	1	$2\pi/M$	0	[0; —; 2, 14], [0; —; 4, 12], [0; —; 6, 10]	3	3
q^9	2	0	0	1	$2\pi/M$	0	[0; —; 2, 16], [0; —; 4, 14], [0; —; 6, 12], [0; —; 8, 10]	4	4
q^{10}	2	0	0	1	$2\pi/M$	0	[0; —; 2, 18], [0; —; 4, 16], [0; —; 6, 14], [0; —; 8, 12]	4	
	4	0	0	2	$2\pi/M$	$-\pi/M$	[−1, 1; —; 2, 4, 6, 8]	1	5
q^{11}	2	0	0	1	$2\pi/M$	0	[0; —; 2, 20], [0; —; 4, 18], [0; —; 6, 16], [0; —; 8, 14], [0; —; 10, 12]	5	
	4	0	0	2	$2\pi/M$	$-\pi/M$	[−1, 1; —; 2, 4, 6, 10]	1	6
q^{12}	2	0	0	1	$2\pi/M$	0	[0; —; 2, 22], [0; —; 4, 20], [0; —; 6, 18], [0; —; 8, 16], [0; —; 10, 14]	5	
	4	0	0	2	$2\pi/M$	$-\pi/M$	[−1, 1; —; 2, 4, 6, 12], [−1, 1; —; 2, 4, 8, 10]	2	7

[a] Here $m_+ = 2m_{ns} + 3m_- + 4m_{-2s}$, and the momentum ranges are: $-(\pi/M)(m_{-2s} + m_- - 2) \leqslant P_j^+ < \infty$, $-(\pi/M)(m_{-2s} + m_- - 2) \leqslant P_j^{-2s} \leqslant (\pi/M)(m_{-2s} + m_- - 2)$, $-(\pi/M)(m_{ns} + 2m_{-2s} + 2m_- - 1) \leqslant P_j^{ns} \leqslant (\pi/M)(m_{ns} + 2m_{-2s} + 2m_- - 1)$.

Table IX. The First Terms for the Sector of the Partition Function Corresponding to $Q=1$ and $m_- = m_{++} \neq 0$[a]

Order	m_+	m_{-2s}	m_-	m_{ns}	$P_{min}^{+,-2s}$	P_{min}^{ns}	$[P^{ns}; P^{-2s}; P^+]$ (units of π/M)	Shift	N	Tot
q^5	3	0	1	0	$2\pi/M$	—	[—; —; 2, 4, 6]	$-2\pi/M$	1	1
q^6	3	0	1	0	$2\pi/M$	—	[—; —; 2, 4, 8]	$-2\pi/M$	1	1
q^7	3	0	1	0	$2\pi/M$	—	[—; —; 2, 4, 10], [—; —; 2, 6, 8]	$-2\pi/M$	2	2
q^8	3	0	1	0	$2\pi/M$	—	[—; —; 2, 4, 12], [—; —; 2, 6, 10],			
							[—; —; 4, 6, 8]	$-2\pi/M$	3	3
q^9	3	0	1	0	$2\pi/M$	—	[—; —; 2, 4, 14], [—; —; 2, 6, 12],			
							[—; —; 4, 6, 10], [—; —; 2, 8, 10]	$-2\pi/M$	4	4
q^{10}	3	0	1	0	$2\pi/M$	—	[—; —; 2, 4, 16], [—; —; 2, 6, 14],			
							[—; —; 4, 6, 12], [—; —; 2, 8, 12],			
							[—; —; 4, 8, 10]	$-2\pi/M$	5	5
q^{11}	3	0	1	0	$2\pi/M$	—	[—; —; 2, 4, 18], [—; —; 2, 6, 16],			
							[—; —; 4, 6, 14], [—; —; 2, 8, 14],			
							[—; —; 4, 8, 12], [—; —; 6, 8, 10],			
							[—; —; 2, 10, 12]	$-2\pi/M$	7	7
q^{12}	3	0	1	0	$2\pi/M$	—	[—; —; 2, 4, 20], [—; —; 2, 6, 18],			
							[—; —; 4, 6, 16], [—; —; 2, 8, 16],			
							[—; —; 4, 8, 14], [—; —; 6, 8, 12],			
							[—; —; 2, 10, 12], [—; —; 4, 10, 12]	$-2\pi/M$	8	
	5	0	1	1	$2\pi/M$	$-3\pi/M$	[-3; —; 2, 4, 6, 8, 10]	$-3\pi/M$	1	9

[a] Here $m_+ = 2m_{ns} + 3m_- + 4m_{-2s}$, and the momentum ranges are: $-(\pi/M)(m_{-2s}+m_- - 3) \leq P_j^+ < \infty$, $-(\pi/M)(m_{-2s}+m_- - 1) \leq P_j^{-2s} \leq (\pi/M)(m_{-2s}+m_- - 3)$, $-(\pi/M)(m_{ns}+2m_{-2s}+2m_-) \leq P_j^{ns} \leq (\pi/M)(m_{ns}+2m_{-2s}+2m_- - 2)$, and there is a shift of $-(2\pi/M)(\frac{1}{2}m_{ns}+m_{-2s}+m_-)$.

Table X. The First Terms for the Sector of the Partition Function Corresponding to $Q=1$ and $m_- = m_{++} \neq 0$[a]

Order	m_+	m_{-2s}	m_-	m_{ns}	$P^{+;-2s}_{min}$	P^{ns}_{min}	$[P^{ns}; P^{-2s}; P^+]$ (units of π/M)	Shift	N	Tot
q^4	3	0	1	0	0	—	[—; —; 0, 2, 4]	$2\pi/M$	1	1
q^5	3	0	1	0	0	—	[—; —; 0, 2, 6]	$2\pi/M$	1	1
q^6	3	0	1	0	0	—	[—; —; 0, 2, 8], [—; —; 0, 4, 6]	$2\pi/M$	2	2
q^7	3	0	1	0	0	—	[—; —; 0, 2, 10], [—; —; 0, 4, 8],			
							[—; —; 2, 4, 6]	$2\pi/M$	3	3
q^8	3	0	1	0	0	—	[—; —; 0, 2, 12], [—; —; 0, 4, 10],			
							[—; —; 2, 4, 8], [—; —; 0, 6, 8]	$2\pi/M$	4	4
q^9	3	0	1	0	0	—	[—; —; 0, 2, 14], [—; —; 0, 4, 12],			
							[—; —; 2, 4, 10], [—; —; 0, 6, 10],			
							[—; —; 2, 6, 8]	$2\pi/M$	5	5
q^{10}	3	0	1	0	0	—	[—; —; 0, 2, 16], [—; —; 0, 4, 14],			
							[—; —; 2, 4, 12], [—; —; 0, 6, 12],			
							[—; —; 2, 6, 10], [—; —; 4, 6, 8],			
							[—; —; 0, 8, 10]	$2\pi/M$	7	7
q^{11}	3	0	1	0	0	—	[—; —; 0, 2, 18], [—; —; 0, 4, 16],			
							[—; —; 2, 4, 14], [—; —; 0, 6, 14],			
							[—; —; 2, 6, 12], [—; —; 4, 6, 10],			
							[—; —; 0, 8, 12], [—; —; 2, 8, 10],	$2\pi/M$	8	
	5	0	1	1	0	$-\pi/M$	[−1; —; 0, 2, 4, 6, 8]	$3\pi/M$	1	9
q^{12}	3	0	1	0	0	—	[—; —; 0, 2, 20], [—; —; 0, 4, 18],			
							[—; —; 2, 4, 16], [—; —; 0, 6, 16],			
							[—; —; 2, 6, 14], [—; —; 4, 6, 12],			
							[—; —; 0, 8, 14], [—; —; 2, 8, 12],			
							[—; —; 0, 10, 12], [—; —; 4, 8, 10]	$2\pi/M$	10	
	5	0	1	1	0	$-\pi/M$	[−1; —; 0, 2, 4, 6, 10],			
							[1; —; 0, 2, 4, 6, 8]	$3\pi/M$	2	12

[a] Here $m_+ = 2m_{ns} + 3m_- + 4m_{-2s}$, and the momentum ranges are: $-(\pi/M)(m_{-2s}+m_- - 1) \leqslant P_j^+ < \infty$, $-(\pi/M)(m_{-2s}+m_- - 1) \leqslant P_j^{-2s} \leqslant (\pi/M)(m_{-2s}+m_- - 3)$, $-(\pi/M)(m_{ns}+2m_{-2s}+2m_- - 2) \leqslant P_j^{ns} \leqslant (\pi/M)(m_{ns}+2m_{-2s}+2m_-)$. There is an additive shift in the total momentum of $(2\pi/M)(\frac{1}{2}m_{ns}+m_{-2s}+m_-)$.

Finally, we find that

$$\sum_{a=1}^{5} \hat{Z}^{(a)}_{1/15} = \hat{\chi}_{1/15} \tag{4.6}$$

This is a five-sum expression for the character $\hat{\chi}_{1/15}$, where each summand can be expressed in the form (1.2). In addition to this form, one can find the forms listed in Table I for the character $\hat{\chi}_{1/15}$. Again, although all these forms are quite different from that of (1.1), they have been shown to be equal to order q^{200}.

It remains to consider the character $\hat{\chi}_{2/3}$. Here, however, no analysis corresponding to the above five-term sum form is availible. The conjectured forms on lines (12)–(14) of Table I have been verified to order q^{200}.

5. DISCUSSION

The forms of the expressions (1.1) and (1.2) for the characters of the ferromagnetic three-state Potts conformal field theory deserve to be called "different," even though the expressions are equal. The question thus arises as to what is meant by the word different, how many different forms there are, and what their significance is. We know of at least four different forms for the characters of the three-state Potts. One is the Rocha-Caridi form (1.1), the second is the form of Kac and Peterson[25] and Jimbo and Miwa,[26] the third is that of Lepowsky and Primc,[13] and the fourth is the form (1.2). Each of these forms is sufficiently different to warrant a separate discussion.

1. The expression (1.1) for the Virasoro characters, which are[27] branching functions of the coset $(A_1^{(1)})_3 \times (A_1^{(1)})_1/(A_1^{(1)})_4$, is what we refer to as a bosonic sum representation. This stems from the presence of the factor $(q)_\infty^{-1}$, which represents a bosonic partition function and can be understood in terms of the Feigin–Fuchs–Felder construction[28,29] of the Virasoro minimal series[30] $\mathcal{M}(p, p')$ to which the three-state Potts conformal field theory belongs, being $\mathcal{M}(5, 6)$ in this notation.

2. The second form is also a bosonic expression which can be obtained by viewing this conformal field theory as that of $\mathbf{Z}_3$-parafermions,[16] where the characters of the corresponding $\mathbf{Z}_3$-parafermionic algebra are[20] the branching functions of the coset $(A_1^{(1)})_3/U(1)$. Another description of the same conformal field theory is as a minimal model with respect to the W_3 algebra,[31] where the corresponding coset construction is $(A_2^{(1)})_1 \times (A_2^{(1)})_1/(A_2^{(1)})_2$. The latter construction is related by level-rank duality[32] to $(A_1^{(1)})_3/U(1)$, and the branching functions are in fact the same.

They are given by the Hecke indefinite forms of[25,26] (or alternative but very similar sum representations of[33])

$$q^{1/30}b^l_m(q)=\frac{q^{h^l_m}}{(q)^2_\infty}\Bigg[\Big(\sum_{s\geq 0}\sum_{n\geq 0}-\sum_{s<0}\sum_{n<0}\Big)\times(-1)^s q^{s(s+1)/2+(l+1)n+(l+m)s/2+5(n+s)n}+\Big(\sum_{s>0}\sum_{n\geq 0}-\sum_{s<0}\sum_{n<0}\Big)\times(-1)^s q^{s(s+1)/2+(l+1)n+(l-m)s/2+5(n+s)n}\Bigg] \tag{5.1}$$

where the h^l_m are

$$h^l_m=\frac{l(l+2)}{20}-\frac{m^2}{12} \tag{5.2}$$

Here $l=0, 1, 2$, $l-m$ is even, and the formulas are valid for $|m|\leqslant l$, while for $|m|>l$ one uses the symmetries

$$b^l_m=b^l_{-m}=b^l_{m+6}=b^{3-l}_{3-m} \tag{5.3}$$

The partition function (1.5) is expressed as a diagonal bilinear form in terms of the b^l_m, through

$$\chi_0+\chi_3=b^0_0,\qquad \chi_{2/5}+\chi_{7/5}=b^2_0,\qquad \chi_{1/15}=b^2_2,\qquad \chi_{2/3}=b^0_2 \tag{5.4}$$

Note that two of the b^l_m split into a sum of a pair of Virasoro characters, corresponding to a more refined splitting of the spectrum of the Hamiltonian into various sectors. Also, the expressions (1.1) have only one factor of $(q)^{-1}_\infty$, while the ones in (5.1) have two. Thus, whereas (1.1) can be said to be based on one boson, (5.1) is based on two bosons.

3. The third form is a fermionic sum representation for the branching functions b^l_m which was obtained by Lepowsky and Primc[13]:

$$q^{1/30}b^l_{2Q-l}(q)=q^{l(2-l)/16}\sum_{\substack{m_1,m_2=0\\ m_1-m_2\equiv Q\ (\mathrm{mod}\ 3)}}^{\infty}\frac{q^{\mathbf{m}C^{-1}_{A_2}\mathbf{m}'+L_l(\mathbf{m})}}{(q)_{m_1}(q)_{m_2}} \tag{5.5}$$

where

$$C_{A_2}=\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}$$

is the Cartan matrix of the Lie algebra A_2, and $L_0(\mathbf{m})=0$, $L_1(\mathbf{m})=(2m_1+m_2)/3$, $L_2(\mathbf{m})=(m_1+2m_2)/3$. This expression can be interpreted[34] in terms of two C-conjugate fermionic quasiparticles carrying $\mathbf{Z}_3$ charges ± 1, both having macroscopic momentum ranges. For example, for $l=0$ these ranges are

$$\frac{2\pi}{M}\left[\frac{1}{2}+\frac{1}{2}\left(\frac{m_1+2m_2}{3}\right)\right]\leqslant P_j^1<\infty, \qquad \frac{2\pi}{M}\left[\frac{1}{2}+\frac{1}{2}\left(\frac{2m_1+m_2}{3}\right)\right]\leqslant P_j^2<\infty \tag{5.6}$$

where the P_j^α $(j=1,..., m_\alpha)$ are chosen from a grid with spacing $2\pi/M$.

4. The fourth form is the fermionic sum representation (1.2), which has one genuine quasiparticle with a macroscopic momentum range and two "ghost" quasiparticles, whose momenta are limited to a microscopic range, e.g., Eqs. (3.1)–(3.3).

The existence of different fermionic sum representations for characters is closely related to the fact that one conformal field theory may have several integrable perturbations, characterized by the conformal dimensions $(\Delta, \bar{\Delta})$ of certain perturbing relevant operators. In ref. 34 this observation was made in connection with the representations of the critical Ising characters as related to either (i) the coset $(A_1^{(1)})_1\times(A_1^{(1)})_1/(A_1^{(1)})_2$, where the character formulas are written in terms of a single quasiparticle and the associated perturbation is by the (1/2, 1/2) operator, or (ii) the coset $(E_8^{(1)})_1\times(E_8^{(1)})_1/(E_8^{(1)})_2$, which has eight quasiparticles and is associated with the (1/16, 1/16) perturbation.[35] In each case the perturbation can be thought of as giving masses to the fermionic quasiparticles.

A similar discussion can be given for the two different fermionic representations (5.5) and (1.2) of the critical three-state Potts model. Consider first (5.5), which was interpreted as having two fermionic quasiparticles of $\mathbf{Z}_3$ charge ± 1. This is to be compared with the (2/5, 2/5) S_3-symmetric perturbation of the three-state Potts conformal field theory, which was argued in ref. 36 to be integrable and to have a spectrum which consists of a $\mathbf{Z}_3$-doublet of massive particles, whose scattering is described by the factorizible S-matrix found in ref. 37. Here the perturbation can again be thought of as giving mass to the two fermionic quasiparticles. This perturbation is also to be compared with the massive $r=5$ RSOS model (or hard squares with diagonal interactions),[38] whose spectrum on the lattice[6] in regime II consists of two excitations with $\mathbf{Z}_3$ charge ± 1.

In the same spirit it is natural to associate the fermionic sum representation (1.2) with the C-even $(2/3, 2/3)^+$ perturbation. This subleading magnetic perturbation breaks the S_3 symmetry down to $\mathbf{Z}_2$. The related statistical mechanics system is the $N=3$ model of Kashiwara and Miwa[39] (also given as the D_4 model of Pasquier,[40] obtained from the $r=6$ RSOS model of Andrews *et al.*[41] by an orbifold construction[42]). In the notation of refs. 40 and 41 the lattice models are to be considered in the regimes III–IV. The perturbed conformal field theory is the $p=5$ case of the $(\Delta_{1,3}, \Delta_{1,3})$-perturbed minimal models $\mathcal{M}(p, p+1)$, which have been discussed in refs. 35 and 43–52, where it is seen that the sign of the coupling to the perturbing operator leads to qualitatively very different effects.

In the case of negative coupling constant (using the conventions of ref. 35) the perturbed theory becomes massive. This is to be compared to the massive regime III of the models of refs. 39–41, where excitation energies have been computed in ref. 53. We interpret this direction of the perturbation as giving mass to the quasiparticle m_1 of (1.2).

The more interesting case is the one where the coupling constant is positive. Now the integrable perturbed conformal field theory remains massless,[35,48] even though scale invariance is broken, and flows[49,51,52,54] from the three-state Potts conformal field theory of central charge 4/5 to the conformal field theory of the tricritical Ising model of central charge 7/10. This suggests an interpretation in terms of the representation (1.2), where we note that under the restriction to the sector where there are no "ghost" excitations of type m_3 the fermionic representations for the three-state Potts characters reduce to fermionic representations[18] for the characters of the tricritical Ising model. Specifically, restricting the summation in (1.2) by setting $m_3=0$, we find that the formulas corresponding to lines (1), (2), (5), (8), (10), and (13) of Table I reduce to expressions for the $c=7/10$ Virasoro characters $\hat{\chi}_\Delta$ with $\Delta=0$, 7/16, 3/2, 3/80, 1/10, and 3/5, respectively. The crucial point making this possible is the fact that (four times) the quadratic form in the fermionic sum representations of the $c=7/10$ characters is the Cartan matrix of A_2, which is precisely the minor, obtained by omitting the last row and column, of the quadratic form C_{A_3} in (1.2). More generally, we find from ref. 18 that the fermionic form of the characters of the unitary minimal model $\mathcal{M}(p, p+1)$ with one quasiparticle and $p-3$ "ghosts" reduces to character formulas for $\mathcal{M}(p-1, p)$ when the last ghost is omitted, the corresponding massless flows being the ones discussed in refs. 43, 44, and 48.

APPENDIX. LOGARITHMIC BETHE EQUATIONS

We recall here some results[21] concerning the classification of the solutions of the Bethe equations corresponding to the eigenvalues of the Hamiltonian (2.1).

Not all the roots λ_j^α [cf. Eq. (2.7)] in a given solution of the Bethe equations (2.5) are independent of one another, and in order to discuss the relations between them we introduce the logarithmic Bethe equations. By taking the logarithm of the Bethe equations (2.5), we can classify the sets $\{\lambda_j^\alpha\}$ more easily. Doing this introduces integers or half-integers associated with the choice of branch of the logarithm. The equations for the complex pairs are first multiplied together. After taking the logarithm, we obtain five sets of equations, one for each class of roots, referred to as the logarithmic Bethe equations:

$$f_\alpha \frac{2\pi}{M} I_j^\alpha = t_\alpha(\lambda_j^\alpha) - \frac{1}{M} \sum_{\beta = \pm, \pm 2s, ns} \sum_{k=1}^{m_\beta} \Theta_{\alpha\beta}(\lambda_j^\alpha - \lambda_k^\beta),$$
$$\alpha \in \{+, -, 2s, -2s, ns\} \tag{A.1}$$

where $f_\alpha = 2$ for $\alpha = ns$ and is 1 otherwise[4] and where the functions $\Theta_{\alpha\beta}$ and t_α are defined as follows. Let $s_\alpha(\lambda) \equiv \sinh(i\alpha - \lambda)/\sinh(i\alpha + \lambda)$; then

$$t_\alpha(\lambda_j^\alpha) = \begin{cases} -2i \ln(\pm s_{\pi/12}(\lambda_j^\pm)) & \alpha = \pm \\ -2i \ln(s_{\pi/12}(\lambda_j^\alpha)\, s_{\pi/12}(\lambda_j^{\alpha *})) & \alpha = \pm 2s, ns \end{cases} \tag{A.2}$$

$$\Theta_{\alpha\beta}(\lambda_j^\alpha - \lambda_k^\beta) = \begin{cases} -i \ln(\varepsilon_{\alpha,\beta} s_{\pi/3}(\lambda_j^\alpha - \lambda_k^\beta)) & \alpha, \beta = \pm \\ -i \ln(\varepsilon_{\alpha,\beta} s_{\pi/3}(\lambda_j^\alpha - \lambda_k^\beta)\, s_{\pi/3}(\lambda_j^\alpha - \lambda_k^{\beta *})) & \alpha = \pm, \beta = \pm 2s, ns \\ -i \ln(\varepsilon_{\alpha,\beta} s_{\pi/3}(\lambda_j^\alpha - \lambda_k^\beta)\, s_{\pi/3}(\lambda_j^{\alpha *} - \lambda_k^\beta)) & \alpha = \pm 2s, ns, \beta = \pm \\ -i \ln(\varepsilon_{\alpha,\beta} s_{\pi/3}(\lambda_j^\alpha - \lambda_k^\beta)\, s_{\pi/3}(\lambda_j^\alpha - \lambda_k^{\beta *}) & \\ \quad \times s_{\pi/3}(\lambda_j^{\alpha *} - \lambda_k^\beta)\, s_{\pi/3}(\lambda_j^{\alpha *} - \lambda_k^{\beta *})) & \alpha, \beta = \pm 2s, ns \end{cases} \tag{A.3}$$

where the symmetric tensor $\varepsilon_{\alpha,\beta}$ is defined by $\varepsilon_{+,-} = \varepsilon_{-,2s} = \varepsilon_{+,-2s} = \varepsilon_{2s,2s} = \varepsilon_{-2s,-2s} = -1$ and the other $\varepsilon_{\alpha,\beta}$ are 1. The $\varepsilon_{\alpha,\beta}$ is chosen so that $\Theta_{\alpha\beta}(\lambda^\alpha - \lambda^\beta) = 0$ when $\Re\mathrm{e}\, \lambda^\alpha = \Re\mathrm{e}\, \lambda^\beta$. All logarithms in (A.2) and (A.3) are chosen such that $-\pi < \Im\mathrm{m} \ln z < \pi$. Each set of (half-) integers $\{I_j^\alpha\}$ uniquely specifies a set of roots $\{\lambda_j^\alpha\}$. Note that the sets contain *either* integers *or* half-integers, depending on m_α.

For the sector $Q = 0$, there is a restriction on the number m_+ of the form

$$m_+ = 2n_{ns} + 3m_- + 4m_{-2s} \tag{A.4}$$

[4] Note that the factor f_{ns} was not present in the definitions of ref. 21. This amounts to a redefinition of the integers I^{ns} discussed there.

In addition, the total number of roots is $2M$ [see Eq. (2.4)], so that

$$M = m_{2s} + 2m_{ns} + 3m_{-2s} + 2m_{-} \tag{A.5}$$

For the sector $Q = \pm 1$, we define the number m_{++}, which has the property that $m_{-} - m_{++} = 0, \pm 1$. For this sector we have the sum rule

$$m_{+} = 2m_{ns} + m_{-} + 2m_{++} + 4m_{-2s} \tag{A.6}$$

and since the total number of roots is $2(M-1)$ [see Eq. (2.4)], we have

$$M - 1 = m_{2s} + 2m_{ns} + 3m_{-2s} + m_{-} + m_{++} \tag{A.7}$$

The (half-) integers in Eq. (A.1) are not all independent, as the sets $\{I_j^{2s}\}$ and $I_j^{-}\}$ are completely determined from the sets $\{I_j^{+}\}$ and $\{I_j^{-2s}\}$, respectively. The ground state of the ferromagnetic chain consists of a sea of $2s$-excitations, that is, the integers $\{I_j^{2s}\}$ fill a symmetric interval about zero, and all other sets of integers are null sets. Therefore, for convenience, we take the sets $\{I_j^{+}\}$, $\{I_j^{-2s}\}$, and $\{I_j^{ns}\}$ to be the independent sets in discussing the ferromagnetic case. Those (half-) integers are then freely chosen from the intervals

$$\begin{gathered}
-\tfrac{1}{2}[M + m_{-} + m_{-2s} - a_l^{(1)}] \leqslant I_j^{+} \leqslant +\tfrac{1}{2}[M + m_{-} + m_{-2s} - a_r^{(1)}] \\
-\tfrac{1}{2}[m_{-} + m_{-2s} - a_r^{(1)}] \leqslant I_j^{-2s} \leqslant +\tfrac{1}{2}[m_{-} + m_{-2s} - a_l^{(1)}] \\
-\tfrac{1}{2}[2m_{-} + 2m_{-2s} + m_{ns} - a_l^{(2)}] \leqslant I_j^{ns} \leqslant +\tfrac{1}{2}[2m_{-} + 2m_{-2s} + m_{ns} - a_r^{(2)}]
\end{gathered} \tag{A.8}$$

with a fermionic exclusion rule: $I_j^{\alpha} \neq I_k^{\alpha}$ for $j \neq k$. The numbers a_l and a_r depend on the sector in question. For the $Q = 0$ sector,

$$a_l^{(1)} = a_r^{(1)} = a_l^{(2)} = a_r^{(2)} = 1 \qquad \text{for} \quad Q = 0 \tag{A.9}$$

In the $Q = \pm 1$ sectors, there are five separate subsectors to be considered, depending on the value of m_{++} introduced above:

$$\text{for} \quad m_{-} - m_{++} = +1: \qquad a_l^{(1)} = a_r^{(1)} = 3, \quad a_l^{(2)} = a_r^{(2)} = 3 \tag{A.10}$$

$$\text{for} \quad m_{-} - m_{++} = -1: \qquad a_l^{(1)} = a_r^{(1)} = 1, \quad a_l^{(2)} = a_r^{(2)} = -1 \tag{A.11}$$

$$\text{for} \quad m_{-} = m_{++} = 0: \qquad a_l^{(1)} = a_r^{(1)} = 2, \quad a_l^{(2)} = a_r^{(2)} = 1 \tag{A.12}$$

$$\text{for} \quad m_{-} = m_{++} \neq 0: \qquad a_l^{(1)} = 3, \quad a_r^{(1)} = 1, \quad a_l^{(2)} = 0, \quad a_r^{(2)} = 2 \tag{A.13}$$

$$\text{for} \quad m_{-} = m_{++} \neq 0: \qquad a_l^{(1)} = 1, \quad a_r^{(1)} = 3, \quad a_l^{(2)} = 2, \quad a_r^{(2)} = 0 \tag{A.14}$$

The last two sectors correspond to two degenerate sets of energy eigenvalues.

The total momentum of each state is determined from Eq. (2.6), and can be expressed in terms of $\{I_j^\alpha\}$ using the logarithmic Bethe equations (A.1). Taking the logarithm of Eq. (2.6) and using the definitions (A.2), we can write the total momentum as

$$P \equiv \tfrac{1}{2} \sum_{j=1}^{m_+} t_+(\lambda_j^+) + \tfrac{1}{2} \sum_{j=1}^{m_-} [t_-(\lambda_j^-) + 2\pi] + \tfrac{1}{2} \sum_{j=1}^{m_{2s}} t_{2s}(\lambda_j^{2s})$$
$$+ \tfrac{1}{2} \sum_{j=1}^{m_{-2s}} t_{-2s}(\lambda_j^{-2s}) + \tfrac{1}{2} \sum_{j=1}^{m_{ns}} t_{ns}(\lambda_j^{ns}) \pmod{2\pi} \tag{A.15}$$

We sum the logarithmic Bethe equations (A.1) over j and α. The sum over the functions $\Theta_{\alpha\beta}$ vanishes since they are odd functions. We are left with a sum over the integers:

$$P \equiv \frac{2\pi}{M} \left[\frac{1}{2} \sum_{j=1}^{m_+} I_j^+ + \frac{1}{2} \sum_{j=1}^{m_-} (I_j^- + M) + \frac{1}{2} \sum_{j=1}^{m_{2s}} I_j^{2s} \right.$$
$$\left. + \frac{1}{2} \sum_{j=1}^{m_{-2s}} I_j^{-2s} + \sum_{j=1}^{m_{ns}} I_j^{ns} \right] \pmod{2\pi} \tag{A.16}$$

In order to express the momentum in terms of three independent sets of integers, we note that for the sector $Q = 0$, as well as for the sectors corresponding to Eqs. (A.10)–(A.12), where the (half-) integers are chosen from a symmetric interval about zero, the two sets of (half-) integers $\{I_j^+\}$ and $\{-I_j^{2s}\}$ fill this interval, and similarly for the sets $\{I_j^-\}$ and $\{-I_j^{-2s}\}$. Therefore,

$$\sum_{j=1}^{m_{\pm 2s}} I_j^{\pm 2s} - \sum_{j=1}^{m_\pm} I_j^\pm = 0 \tag{A.17}$$

and the total momentum of a state may be written [using $m_+ \equiv m_- \pmod 2$] as

$$P \equiv \frac{2\pi}{M} \left(\sum_{j=1}^{m_+} \bar{I}_j^+ + \sum_{j=1}^{m_{-2s}} I_j^{-2s} + \sum_{j=1}^{m_{ns}} I_j^{ns} \right) \pmod{2\pi} \tag{A.18}$$

where $\bar{I}_j^+ = I_j^+ + M/2$.

However, for the sectors corresponding to Eqs. (A.13)–(A.14) there is an additional term involved, since the integer ranges are not symmetric about zero, and there is an offset between the sets $\{I_j^\pm\}$ and $\{-I_j^{\pm 2s}\}$. In fact, for the sector (A.13) the following relation between the integers holds[21]:

$$I_j^{2s(\mathrm{h})} = -I_j^+ + \tfrac{1}{2}, \qquad I_j^{-(\mathrm{h})} = -I_j^{-2s} - \tfrac{1}{2} \tag{A.19}$$

where the superscript (h) refers to "holes," namely the (half-) integers missing from the set $\{I_j^\alpha\}$. The number of 2s-holes is m_+, and the number of "$-$"-holes is m_{-2s}. The ranges of integers are chosen such that

$$\sum_{j=1}^{m_{2s}} I_j^{2s} + \sum_{j=1}^{m_+} I_j^{2s(\mathrm{h})} = 0 \tag{A.20}$$

that is, the I_j^{2s} are chosen from a symmetric range. This is not the case for the I_j^-, which are chosen from the range

$$-\tfrac{1}{2}(m_- + m_{-2s}) \leqslant I_j^- \leqslant \tfrac{1}{2}(m_- + m_{-2s} - 2) \tag{A.21}$$

so that

$$\sum_{j=1}^{m_-} I_j^- + \sum_{j=1}^{m_{-2s}} I_j^{-(\mathrm{h})} = -\tfrac{1}{2}(m_- + m_{-2s}) \tag{A.22}$$

Putting Eqs. (A.19)–(A.24) together, we find that for this sector

$$P \equiv \frac{2\pi}{M}\left[\sum_{j=1}^{m_+} \bar{I}_j^+ + \sum_{j=1}^{m_{-2s}} I_j^{-2s} + \sum_{j=1}^{m_{ns}} I_j^{ns} - \left(\frac{1}{2} m_{ns} + m_- + m_{-2s}\right)\right] \pmod{2\pi} \tag{A.23}$$

For the sector corresponding to Eq. (A.14) we have

$$I_j^{2s(\mathrm{h})} = -I_j^+ - \tfrac{1}{2}, \qquad I_j^{-(\mathrm{h})} = -I_j^{-2s} + \tfrac{1}{2} \tag{A.24}$$

Equation (A.20) still holds, but the range of I_j^- is now such that

$$\sum_{j=1}^{m_-} I_j^- + \sum_{j=1}^{m_{-2s}} I_j^{-(\mathrm{h})} = \tfrac{1}{2}(m_- + m_{-2s}) \tag{A.25}$$

Therefore the total momentum in this sector is found to be

$$P \equiv \frac{2\pi}{M}\left[\sum_{j=1}^{m_+} \bar{I}_j^+ + \sum_{j=1}^{m_{-2s}} I_j^{-2s} + \sum_{j=1}^{m_{ns}} I_j^{ns} + \left(\frac{1}{2} m_{ns} + m_- + m_{-2s}\right)\right] \pmod{2\pi} \tag{A.26}$$

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